

The Erdős and Guy's conjectured equality on the crossing number of hypercubes *

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Abstract

Let Q_n be the n -dimensional hypercube, and let $\text{cr}(Q_n)$ be the *crossing number* of Q_n . A long-standing conjecture proposed by Erdős and Guy in 1973 is the following equality: $\text{cr}(Q_n) = \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$. In this paper, we construct a drawing of Q_n with less crossings, which implies $\text{cr}(Q_n) \leq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$.

Keywords: *Drawing; Crossing number; Hypercube; Erdős-Guy's conjecture*

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The *crossing number* of a graph G , denoted $\text{cr}(G)$, is the minimum possible number of edge crossings in a drawing of G in the plane. The notion of crossing number is a central one for Topological Graph Theory and has been studied extensively by mathematicians including Erdős, Guy, Harary, Turán and Tutte, et al. (see [3, 4, 9–11]).

However, the investigation on the crossing number of graphs is an extremely difficult problem. In 1973, Erdős and Guy [3] wrote, “*Almost all questions that one can ask about crossing numbers remain unsolved.*” Actually, Garey and Johnson [7] proved that computing the crossing number

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is NP-complete. In this field, Eggleton and Guy [2] in 1970 constructed a drawing of Q_n with exactly $\frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$ crossings, which implies

$$\text{cr}(Q_n) \leq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}, \quad (1)$$

where Q_n denotes the n -dimensional hypercube.

We still quote: “*but a gap has been found in the description of the construction, so this must also remain a conjecture. We again conjecture equality in (1).*” (P. Erdős and R.K. Guy [3])

For a long time, the above Erdős and Guy’s conjectured *equality* was inclined to be “trustable”, even though to prove or disprove it seems impossible. The past results on $\text{cr}(Q_n)$, including $\text{cr}(Q_3) = 0$ (trivial), $\text{cr}(Q_4) = 8$ (see [1]), and the obtained best drawings for Q_n with $n = 5, 6, 7, 8$ (see [5], [8]) seem to support *the equality*.

Far beyond these, very recently in 2008, Faria, Figueiredo, Sýkora and Vřto [6] announced a drawing for which the number of crossings coincides with $\frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$, which would imply the *inequality* of (1).

However, we noticed a hiatus in the construction of [6]. In more specific terms, Property 3 in [6], one important condition to make sure that the construction can be implemented inductively, seems to fail, so that the *Meshes* employed by them to decrease the crossings in drawings can not be consistent “matched” for $n \geq 9$. Nevertheless, with regard to this problem, the greatest contribution is still from the four authors, Faria, Figueiredo, Sýkora and Vřto.

Inspired by their original work, we employ a new strategy to construct a drawing of the hypercube Q_n with less crossings than the values conjectured by Erdős and Guy, which implies that

$$\text{cr}(Q_n) \not\geq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}.$$

In particular, we prove the following upper bound for the crossing number of the hypercubes.

Theorem 1.1.

$$\text{cr}(Q_n) \leq \begin{cases} \frac{139}{896}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2} + \frac{4}{7} \cdot 2^{3\lfloor \frac{n}{2} \rfloor - n}, & \text{if } 5 \leq n \leq 10; \\ \frac{26695}{172032}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2} - \frac{n^2+2}{3} \cdot 2^{n-2} + \frac{4}{7} \cdot 2^{3\lfloor \frac{n}{2} \rfloor - n}, & \text{if } n \geq 11. \end{cases}$$

2 Preliminaries

A drawing of G is said to be a *good* drawing, provided that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. It is well known that the crossing number of a graph is attained only in *good* drawings of the graph. So, we always assume that all drawings throughout this paper are good drawings. Let D be a good drawing of the graph G , and let A and B be two disjoint subsets of $E(G)$. In the drawing D , the number of the crossings formed by an edge in A and another edge in B is denoted by $\nu_D(A, B)$. The number of the crossings that involve a pair of edges in A is denoted

by $\nu_D(A)$. In what follows, $\nu_D(E(G))$ is abbreviated to $\nu(D)$ when it is unambiguous. Let u be a vertex of G , and let U be a vertex subset of G . We define $\mathcal{I}(u)$ to be the edge subset of $E(G)$ in which every edge is incident to u . Let $\mathcal{I}(U) = \bigcup_{u \in U} \mathcal{I}(u)$, and let $\partial(U) = \mathcal{I}(U) \setminus E(U)$. Suppose that the graph G is drawn in the 2-dimensional Euclidean plane $\mathbb{R} \times \mathbb{R}$. By X_u and Y_u we denote the X and Y -coordinates of u in $\mathbb{R} \times \mathbb{R}$.

The n -dimensional hypercube Q_n is a graph with the vertex set $V(Q_n) = \{d_1 d_2 \cdots d_n : d_i \in \{0, 1\}, i = 1, 2, \dots, n\}$, for which any two vertices $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$ are adjacent if and only if there exists a unique $i \in [1, n]$ such that $a_i \neq b_i$. In particular, if the unique $i \in [1, n]$ with $a_i \neq b_i$ is equal to $n - 1$, we denote $b = \widehat{a}$, and conversely, $a = \widehat{b}$. For any binary string $x_1 x_2 \cdots x_t$, we denote $a^{(x_1 x_2 \cdots x_t)} = a_1 a_2 \cdots a_n x_1 x_2 \cdots x_t$ to be the vertex of Q_{n+t} .

- For the clearness of composition, in the rest of this paper, any vertex $a = a_1 a_2 \cdots a_n \in V(Q_n)$ in figures will be represented by the corresponding decimal number $2^{n-1}a_1 + 2^{n-2}a_2 + \cdots + 2^0 a_n$.

Now we shall define some structures, which will be used in characterizing the constructions of later drawings. Let $\delta \in \{2, 4\}$ and $k \in \mathbb{N}$, and let P_1, \dots, P_δ be δ vertices drawn in the 2-dimensional Euclidean plane $\mathbb{R} \times \mathbb{R}$ with

$$X_{P_1} = X_{P_2} = \cdots = X_{P_\delta}$$

and

$$Y_{P_1} > Y_{P_2} > \cdots > Y_{P_\delta}.$$

The structure ML_k^δ is defined as follows. Draw k groups of non-vertical straight lines in the left plane of the line $x = X_{P_1}$, and $k - 1$ groups of non-vertical straight lines in the right plane of the line $x = X_{P_1}$, such that each group is consisting of δ straight lines which are extending from vertices P_1, \dots, P_δ respectively and parallel each other. For all $i \in [1, \delta - 1]$, join P_i and P_{i+1} in a straight line, i.e., the edge $P_i P_{i+1}$ is drawn precisely on the line $x = X_{P_1}$. Moreover, if $\delta = 4$, we join P_1 and P_4 in an arc drawn on the right of the line $x = X_{P_1}$.

The structures \widetilde{ML}_k^T is obtained from ML_k^4 by rotating the straight line which is extending from P_4 and belongs to the top group on the left plane of $x = X_{P_1}$, and rotating the straight line which is extending from P_3 and belongs to the bottom group on the right plane of $x = X_{P_1}$.

The structures \widetilde{ML}_k^B is obtained from ML_k^4 by rotating the straight line which is extending from P_1 and belongs to the bottom group on the left plane of $x = X_{P_1}$, and rotating the straight line which is extending from P_2 and belongs to the top group on the right plane of $x = X_{P_1}$.

The structures MR_k^δ , \widetilde{MR}_k^T and \widetilde{MR}_k^B are drawn to be reflections of ML_k^δ , \widetilde{ML}_k^T and \widetilde{ML}_k^B respectively, with the “mirror” $x = X_{P_1}$.

For the convenience of the readers, we give the examples of the above structures with $k = 4$ in Figure 2.1 and Figure 2.2. Moreover, in what follows, ML_k^4 and MR_k^4 are abbreviated to ML_k and MR_k respectively, when it is unambiguous.

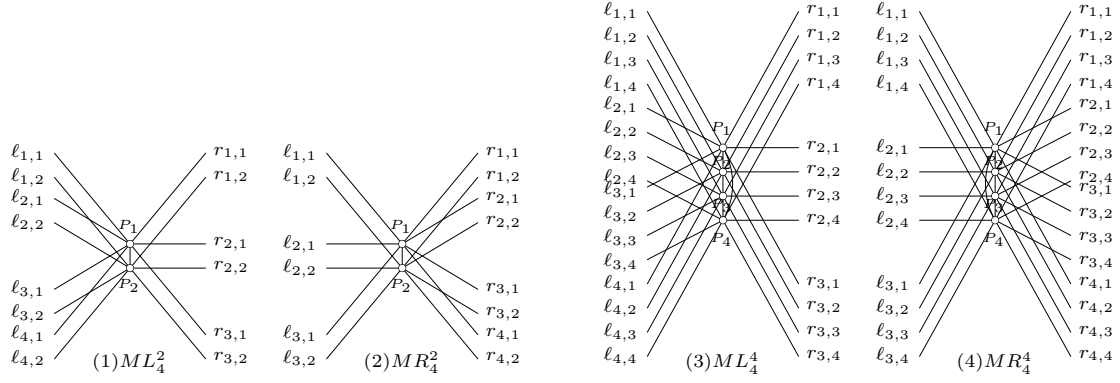


Figure 2.1: Drawings of ML_4^2 , MR_4^2 , ML_4^4 and MR_4^4

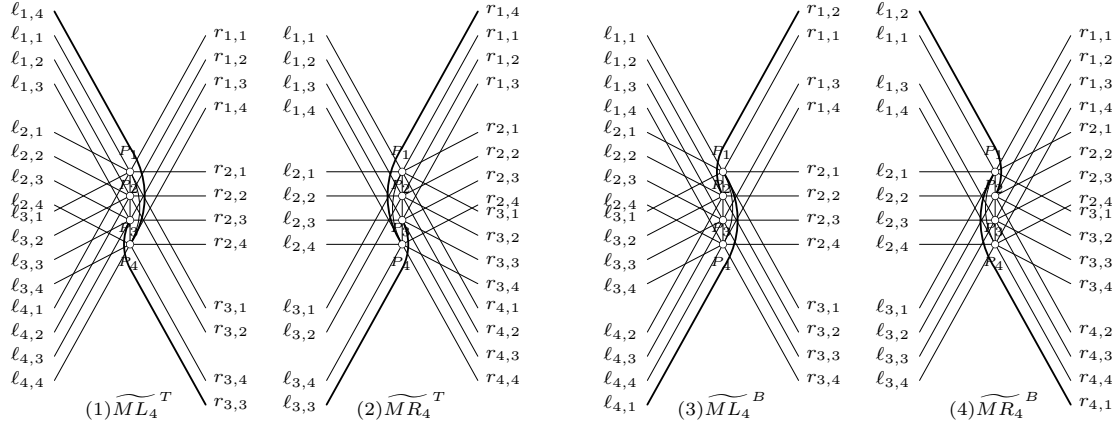


Figure 2.2: Drawings of \widetilde{ML}_4^T , \widetilde{MR}_4^T , \widetilde{ML}_4^B and \widetilde{MR}_4^B ,

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by constructing a drawing, denoted Γ_n , of Q_n with the desired crossings for every integer $n \geq 5$. The constructions of the drawing Γ_n are different according to $n \equiv 1 \pmod{2}$ or not. Hence, we shall introduce the constructions of Γ_n in Subsection 3.1 and Subsection 3.2 accordingly. In Subsection 3.3, we need to verify that the constructed drawing Γ_n has the desired number of crossings.

3.1 Construction of Γ_n for odd n

- Throughout this subsection, we always use n as an odd integer no less than 5.

Note first that given any $n \geq 5$, all the drawings of Q_n in this subsection will have the same locations of vertices in $\mathbb{R} \times \mathbb{R}$. Hence, we shall denote without loss of generality that \mathcal{D}_n to be an arbitrary drawing of Q_n , which shares the following inductive rule for the arrangements of vertices:

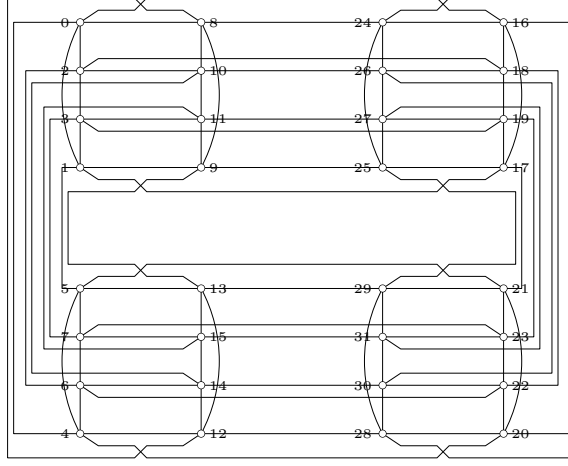


Figure 3.1: The drawing Γ_5 with 56 crossings

For $n = 5$, the locations of vertices in \mathcal{D}_5 are shown as in Figure 3.1 for Γ_5 , where

$$\{(X_u, Y_u) : u \text{ is a vertex in } \mathcal{D}_5\} = \{-2, -1, 1, 2\} \times \{-4, -3, -2, -1, 1, 2, 3, 4\}.$$

Suppose $n > 5$. Fix a large positive integer \mathcal{N} . Take an arbitrary vertex u in \mathcal{D}_{n-2} . The vertices $u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)}$ in \mathcal{D}_n will be located in $\mathbb{R} \times \mathbb{R}$ such that

$$X_{u^{(00)}} = X_{u^{(10)}} = X_{u^{(11)}} = X_{u^{(01)}} = X_u \quad (2)$$

and

$$\begin{cases} Y_{u^{(00)}} = Y_u; \\ Y_{u^{(10)}} = Y_u + \frac{Y_{\widehat{u}} - Y_u}{\mathcal{N}}; \\ Y_{u^{(11)}} = Y_u + 2 \cdot \frac{Y_{\widehat{u}} - Y_u}{\mathcal{N}}; \\ Y_{u^{(01)}} = Y_u + 3 \cdot \frac{Y_{\widehat{u}} - Y_u}{\mathcal{N}}. \end{cases} \quad (3)$$

We also denote by $\mathcal{P}_1(u), \mathcal{P}_2(u), \mathcal{P}_3(u), \mathcal{P}_4(u)$ the four vertices of $\{u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)}\}$ given as (2) and (3), with always

$$Y_{\mathcal{P}_1(u)} > Y_{\mathcal{P}_2(u)} > Y_{\mathcal{P}_3(u)} > Y_{\mathcal{P}_4(u)},$$

equivalently,

$$(\mathcal{P}_1(u), \mathcal{P}_2(u), \mathcal{P}_3(u), \mathcal{P}_4(u)) = \begin{cases} (u^{(00)}, u^{(10)}, u^{(11)}, u^{(01)}), & \text{if } Y_{\widehat{u}} < Y_u; \\ (u^{(01)}, u^{(11)}, u^{(10)}, u^{(00)}), & \text{if } Y_{\widehat{u}} > Y_u. \end{cases} \quad (4)$$

By (2), (3) and the induction on n , we conclude that for any two adjacent vertices u_1 and u_2 in \mathcal{D}_n ,

$$X_{u_1} = X_{u_2} \text{ or } Y_{u_1} = Y_{u_2}, \quad (5)$$

and

$$\text{sgn}(Y_{\widehat{u_2}} - Y_{u_2}) = \begin{cases} -\text{sgn}(Y_{\widehat{u_1}} - Y_{u_1}), & \text{if } X_{u_1} = X_{u_2}; \\ \text{sgn}(Y_{\widehat{u_1}} - Y_{u_1}), & \text{if } Y_{u_1} = Y_{u_2}, \end{cases} \quad (6)$$

and that for any vertex u in \mathcal{D}_n , there exist three vertices, denoted u' , u'' and u''' , in \mathcal{D}_n such that

$$\begin{aligned}(X_{u'}, Y_{u'}) &= (X_u, 5 \cdot \text{sgn}(Y_u) - Y_u), \\ (X_{u''}, Y_{u''}) &= (X_u, -Y_u), \\ (X_{u'''}, Y_{u'''}) &= (-X_u, Y_u),\end{aligned}$$

where $\text{sgn}(\cdot)$ denotes the signum function. In what follows, we denote by $\Theta_1(u)$, $\Theta_2(u)$ and $\Theta_3(u)$ the above three vertices u' , u'' and u''' , respectively.

Meanwhile, it is easy to see the following.

Observation 3.1. *Let u_1, u_2 be two vertices in \mathcal{D}_n . If u_1 and u_2 are adjacent, then so are $\Theta_i(u_1)$ and $\Theta_i(u_2)$ where $i \in [1, 3]$.*

To proceed on, we shall need some technic definitions.

Definition 3.1. *Let u be a vertex and $e \in \mathcal{I}(u)$ in \mathcal{D}_n . We call the edge e*

- **a left arc** with respect to u , if the part of e within the “small neighborhood” of u is drawn on the left of the line $x = X_u$;
- **a right arc** with respect to u , if the part of e within the “small neighborhood” of u is drawn on the right of the line $x = X_u$;
- **a below arc** with respect to u , if the part of e within the “small neighborhood” of u is drawn on the below of the line $y = Y_u$;
- **an upper arc** with respect to u , if the part of e within the “small neighborhood” of u is drawn on the upper of the line $y = Y_u$.

Furthermore, for any vertex u in \mathcal{D}_n , we define

$$\begin{aligned}\mathcal{A}_{\mathcal{D}_n}^{-1}(u) &= \{e \in \mathcal{I}(u) : e \text{ is a left arc with respect to } u \text{ in } \mathcal{D}_n\}, \\ \mathcal{A}_{\mathcal{D}_n}^1(u) &= \{e \in \mathcal{I}(u) : e \text{ is a right arc with respect to } u \text{ in } \mathcal{D}_n\}, \\ \mathcal{A}_{\mathcal{D}_n}^{-2}(u) &= \{e \in \mathcal{I}(u) : e \text{ is a below arc with respect to } u \text{ in } \mathcal{D}_n\}, \\ \mathcal{A}_{\mathcal{D}_n}^2(u) &= \{e \in \mathcal{I}(u) : e \text{ is an upper arc with respect to } u \text{ in } \mathcal{D}_n\}.\end{aligned}$$

Note that there may exist some edges incident to u which are neither left arcs nor right arcs with respect to u , i.e., $\mathcal{A}_{\mathcal{D}_n}^{-1}(u) \cup \mathcal{A}_{\mathcal{D}_n}^1(u)$ is perhaps a proper subset of $\mathcal{I}(u)$. For example, in Figure 3.1 the edge joining vertex “0” and vertex “2” is so with respect to both its ends. Similarly, $\mathcal{A}_{\mathcal{D}_n}^{-2}(u) \cup \mathcal{A}_{\mathcal{D}_n}^2(u)$ is also perhaps a proper subset of $\mathcal{I}(u)$. To characterize more precisely the drawing of any edge within the “small neighborhood” of its ends, we need to introduce the following notations.

Let $\lambda_{\mathcal{D}_n;u} : \mathcal{I}(u) \rightarrow \{J : J \subseteq \{-2, -1, 1, 2\}\}$ be a map such that

$$\lambda_{\mathcal{D}_n;u}(e) = \{j \in \{-2, -1, 1, 2\} : e \in \mathcal{A}_{\mathcal{D}_n}^j(u)\},$$

and let $\mathcal{N}_{\mathcal{D}_n;u} : \mathcal{I}(u) \rightarrow [-|\mathcal{A}_{\mathcal{D}_n}^{-1}(u)|, |\mathcal{A}_{\mathcal{D}_n}^1(u)|] \cup \{n+1\}$ be an injection such that for any edge $e \in \mathcal{I}(u)$,

$$\mathcal{N}_{\mathcal{D}_n;u}(e) \in \begin{cases} [-|\mathcal{A}_{\mathcal{D}_n}^{-1}(u)|, -1], & \text{if } -1 \in \lambda_{\mathcal{D}_n;u}(e); \\ [1, |\mathcal{A}_{\mathcal{D}_n}^1(u)|], & \text{if } 1 \in \lambda_{\mathcal{D}_n;u}(e); \\ \{0\}, & \text{if } \lambda_{\mathcal{D}_n;u}(e) = \{2\}; \\ \{n+1\}, & \text{if } \lambda_{\mathcal{D}_n;u}(e) = \{-2\}, \end{cases}$$

and that all the edges e_1, e_2, \dots, e_n of $\mathcal{I}(u)$ are in clockwise around u in turn with $\mathcal{N}_{\mathcal{D}_n;u}(e_1) < \mathcal{N}_{\mathcal{D}_n;u}(e_2) < \dots < \mathcal{N}_{\mathcal{D}_n;u}(e_n)$.

The following structure will be crucial in this paper to construct the drawing of Q_n with the desired number of crossings.

Definition 3.2. Let $U = \{u_1, u_2, \dots, u_8\}$ be a set of eight vertices in \mathcal{D}_n . We say that the drawing of the edge set $\mathcal{I}(U)$ forms a **Fundamental structure**, provided that the drawing of $\mathcal{I}(U)$ is given as either Diagram (1) or Diagram (2) in Figure 3.2, in particular, with

$$|Y_{u_1}| = |Y_{u_2}| = \dots = |Y_{u_8}|,$$

$$|\partial(U) \cap \mathcal{A}_{\mathcal{D}_n}^{-2}(u_i)| = |\partial(U) \cap \mathcal{A}_{\mathcal{D}_n}^2(u_i)| = \frac{n-3}{2} \quad \text{for all } i \in [1, 8], \quad (7)$$

and no vertex lying in the interior of the cycle C , where C denotes any of the 4-cycles $u_1u_2u_3u_4$, $u_3u_4u_5u_6$, $u_5u_6u_7u_8$ and $u_7u_8u_1u_2$.

Furthermore, if the drawing of the edge set $\mathcal{I}(U)$ in \mathcal{D}_n forms a Fundamental structure, any edge of $E(U)$ is called a **Fundamental edge**.

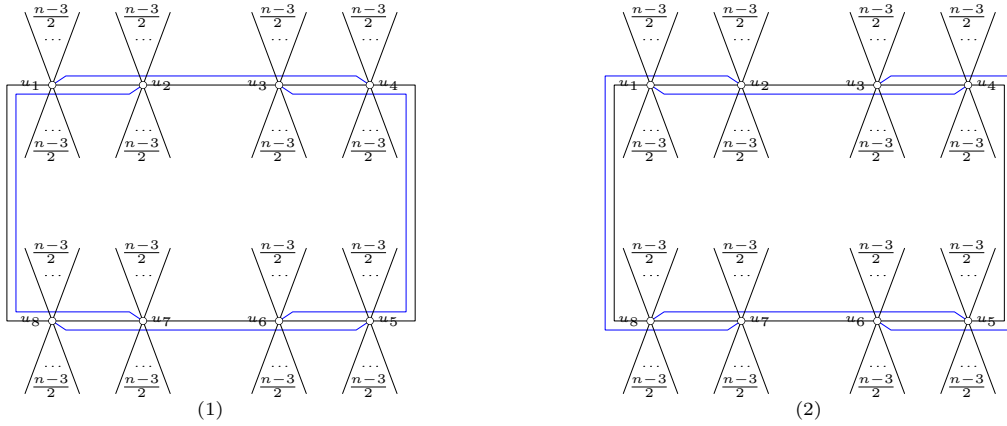


Figure 3.2: Fundamental structures

Definition 3.3. Let $u_0u_1 \dots u_{k-1}$ be a cycle in \mathcal{D}_n . We call $u_0u_1 \dots u_{k-1}$ an **Enclosed cycle** provided that

$$(|\mathcal{A}_{\mathcal{D}_n}^{-1}(u_i)|, |\mathcal{A}_{\mathcal{D}_n}^1(u_i)|) \in \{(\frac{n-1}{2}, \frac{n+1}{2}), (\frac{n+1}{2}, \frac{n-1}{2})\}, \quad (8)$$

$$\begin{aligned} (\mathcal{N}_{\mathcal{D}_n;u_i}(u_{i-1}u_i), \mathcal{N}_{\mathcal{D}_n;u_i}(u_iu_{i+1})) &\in \{(-1, |\mathcal{A}_{\mathcal{D}_n}^1(u_i)|), (1, -|\mathcal{A}_{\mathcal{D}_n}^{-1}(u_i)|), \\ &(|\mathcal{A}_{\mathcal{D}_n}^1(u_i)|, -1), (-|\mathcal{A}_{\mathcal{D}_n}^{-1}(u_i)|, 1)\}, \end{aligned} \quad (9)$$

$$|\mathcal{A}_{\mathcal{D}_n}^{\text{sgn}(\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1}))}(u_i)| = |\mathcal{A}_{\mathcal{D}_n}^{\text{sgn}(\mathcal{N}_{\mathcal{D}_n;u_{i+1}}(u_i u_{i+1}))}(u_{i+1})| \quad (10)$$

and

$$|\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1})| - |\mathcal{N}_{\mathcal{D}_n;u_{i+1}}(u_i u_{i+1})| \begin{cases} \neq 0, & \text{if } X_{u_i} = X_{u_{i+1}}; \\ = 0, & \text{if } Y_{u_i} = Y_{u_{i+1}}, \end{cases} \quad (11)$$

where the subscripts of u are modulo k .

Definition 3.4. Let $\mathcal{C} = u_0 u_1 \cdots u_{k-1}$ be an Enclosed cycle in \mathcal{D}_n . We define two maps $\mathcal{H}_{\mathcal{C}}^-$ and $\mathcal{H}_{\mathcal{C}}^+$ from $V(\mathcal{C})$ to $[1, 4]$ given as

$$(\mathcal{H}_{\mathcal{C}}^-(u_i), \mathcal{H}_{\mathcal{C}}^+(u_i)) = \begin{cases} (2, 1), & \text{if } (|\mathcal{N}_{\mathcal{D}_n;u_i}(u_{i-1}u_i)|, |\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1})|) = (1, \frac{n+1}{2}); \\ (4, 3), & \text{if } (|\mathcal{N}_{\mathcal{D}_n;u_i}(u_{i-1}u_i)|, |\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1})|) = (1, \frac{n-1}{2}); \\ (1, 2), & \text{if } (|\mathcal{N}_{\mathcal{D}_n;u_i}(u_{i-1}u_i)|, |\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1})|) = (\frac{n+1}{2}, 1); \\ (3, 4), & \text{if } (|\mathcal{N}_{\mathcal{D}_n;u_i}(u_{i-1}u_i)|, |\mathcal{N}_{\mathcal{D}_n;u_i}(u_i u_{i+1})|) = (\frac{n-1}{2}, 1), \end{cases}$$

where the subscripts of u are modulo k .

By (5), we can define the following.

Definition 3.5. Let $u_1 u_2$ be an edge in \mathcal{D}_n . We say that the edge $u_1 u_2$ is **Self-symmetric** provided that the following condition holds:

If $X_{u_1} = X_{u_2}$, then the edge $u_1 u_2$ is drawn symmetric with respect to the line $y = \frac{Y_{u_1} + Y_{u_2}}{2}$, i.e., the part of $u_1 u_2$ drawn on the upper is a reflection of the part drawn on the below with the “mirror” $y = \frac{Y_{u_1} + Y_{u_2}}{2}$, in particular,

$$\lambda_{\mathcal{D}_n;u_1}(u_1 u_2) \cap \{-1, 1\} = \lambda_{\mathcal{D}_n;u_2}(u_1 u_2) \cap \{-1, 1\}$$

and

$$\lambda_{\mathcal{D}_n;u_1}(u_1 u_2) \cap \{-2, 2\} = -(\lambda_{\mathcal{D}_n;u_2}(u_1 u_2) \cap \{-2, 2\}).$$

If $Y_{u_1} = Y_{u_2}$, then the edge $u_1 u_2$ is drawn symmetric with respect to the line $x = \frac{X_{u_1} + X_{u_2}}{2}$, i.e., the part of $u_1 u_2$ drawn on the left is a reflection of the part drawn on the right with the “mirror” $x = \frac{X_{u_1} + X_{u_2}}{2}$, in particular,

$$\lambda_{\mathcal{D}_n;u_1}(u_1 u_2) \cap \{-1, 1\} = -(\lambda_{\mathcal{D}_n;u_2}(u_1 u_2) \cap \{-1, 1\})$$

and

$$\lambda_{\mathcal{D}_n;u_1}(u_1 u_2) \cap \{-2, 2\} = \lambda_{\mathcal{D}_n;u_2}(u_1 u_2) \cap \{-2, 2\}.$$

By Observation 3.1, the following definition is well defined.

Definition 3.6. Let $u_1 u_2$ be an edge in \mathcal{D}_n . We say that the edge $u_1 u_2$ is **Conjunct-symmetric** provided that

$$\begin{cases} \lambda_{\mathcal{D}_n;\Theta_1(u_i)}(\Theta_1(u_1)\Theta_1(u_2)) \cap \{-1, 1\} = \lambda_{\mathcal{D}_n;u_i}(u_1 u_2) \cap \{-1, 1\}; \\ \lambda_{\mathcal{D}_n;\Theta_1(u_i)}(\Theta_1(u_1)\Theta_1(u_2)) \cap \{-2, 2\} = -(\lambda_{\mathcal{D}_n;u_i}(u_1 u_2) \cap \{-2, 2\}), \end{cases}$$

$$\begin{cases} \lambda_{\mathcal{D}_n; \Theta_2(u_i)}(\Theta_2(u_1)\Theta_2(u_2)) \cap \{-1, 1\} = \lambda_{\mathcal{D}_n; u_i}(u_1u_2) \cap \{-1, 1\}; \\ \lambda_{\mathcal{D}_n; \Theta_2(u_i)}(\Theta_2(u_1)\Theta_2(u_2)) \cap \{-2, 2\} = -(\lambda_{\mathcal{D}_n; u_i}(u_1u_2) \cap \{-2, 2\}), \end{cases}$$

and

$$\begin{cases} \lambda_{\mathcal{D}_n; \Theta_3(u_i)}(\Theta_3(u_1)\Theta_3(u_2)) \cap \{-1, 1\} = -(\lambda_{\mathcal{D}_n; u_i}(u_1u_2) \cap \{-1, 1\}); \\ \lambda_{\mathcal{D}_n; \Theta_3(u_i)}(\Theta_3(u_1)\Theta_3(u_2)) \cap \{-2, 2\} = \lambda_{\mathcal{D}_n; u_i}(u_1u_2) \cap \{-2, 2\}, \end{cases}$$

for both $i = 1, 2$.

Now we are in a position to introduce the construction of the drawing Γ_n . In general, the constructed drawing Γ_n shall satisfy the following two properties:

- Each edge in Γ_n is Self-symmetric;
- The number of Fundamental structures in Γ_n is $2^{\frac{n-3}{2}}$.

From Figure 3.1, we can verify that the drawing Γ_5 satisfies the above two properties. For any odd number $n \geq 5$, the drawing Γ_{n+2} will be constructed from Γ_n inductively. To get a clearer understanding, we split the process of obtaining Γ_{n+2} from Γ_n into three steps. In the first step, we obtain another drawing of Q_n , denoted Γ_n^* , such that the number of left arcs and the number of right arcs with respect to any vertex u in Γ_n^* are “balanced”, in particular, either

$$(|\mathcal{A}_{\Gamma_n^*}^{-1}(u)|, |\mathcal{A}_{\Gamma_n^*}^1(u)|) = \left(\frac{n-1}{2}, \frac{n+1}{2}\right) \quad (12)$$

or

$$(|\mathcal{A}_{\Gamma_n^*}^{-1}(u)|, |\mathcal{A}_{\Gamma_n^*}^1(u)|) = \left(\frac{n+1}{2}, \frac{n-1}{2}\right). \quad (13)$$

In the second step, we get a drawing of Q_{n+2} , denoted $\tilde{\Gamma}_{n+2}$, by replacing each edge in Γ_n^* , say u_1u_2 , by a bunch of four new edges $u_1^{(00)}u_2^{(00)}, u_1^{(11)}u_2^{(11)}, u_1^{(10)}u_2^{(10)}, u_1^{(01)}u_2^{(01)}$ such that the “bunch” is along the original route of u_1u_2 . In the final step, we adjust the drawings of some edges “locally” or “globally” to decrease the number of crossings in $\tilde{\Gamma}_{n+2}$, and then get the desired drawing Γ_{n+2} .

In the rest of this subsection, we shall characterize each step in detailed.

Step 1. Obtaining Γ_n^* from Γ_n .

In this step, we construct the drawing Γ_n^* satisfying the following four properties:

Property 1. Each edge in Γ_n^* is Self-symmetric;

Property 2. For any vertex u in Γ_n^* , either (12) or (13) holds;

Property 3. The number of Fundamental structures is $2^{\frac{n-3}{2}}$ in Γ_n^* ;

Property 4. For $n \in \{5, 7, 9\}$, there exists a decomposition of $V(Q_n)$ into several disjoint Enclosed cycles, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$, in Γ_n^* such that no Fundamental edge is in any Enclosed cycle.

Since some specific process vary somewhat when $n \in \{5, 7, 9, 11\}$ or $n \geq 13$, we need to distinguish two cases accordingly.

Case 1. Suppose $n \in \{5, 7, 9, 11\}$.

We remark that for this case, the drawing Γ_n^* with the above desired properties is obtained by elaborate attempts instead of a general rule. The adjustments of edges in the process of obtaining Γ_5^* (Γ_7^* , Γ_9^*) from Γ_5 (Γ_7 , Γ_9 correspondingly) can be seen in Figure 3.5, Figure 4.3 and Figure 4.4. In the diagrams for Γ_5^* , Γ_7^* and Γ_9^* , the bold lines mean that the corresponding edges are in some Enclosed cycles. For $n = 11$, since we need only make sure that the drawing Γ_{11}^* has Property 1, Property 2 and Property 3, the process of obtaining Γ_{11}^* from Γ_{11} is much easier. Hence, we omit the explanatory diagrams for $n = 11$ here. Meanwhile, for the simplification of compositions, we illustrate only the adjustments of edges incident to any vertex u with $X_u \in \{-2, -1\}$ and $3 \leq Y_u \leq 4$. The other adjustments of edges in the whole drawing can be easily completed according to the following rules:

Let u_1u_2 be an arbitrary edge in Γ_n^ . Then u_1u_2 is both Self-symmetric and Conjunct-symmetric. Moreover, if u_1u_2 is in some Enclosed cycle, say \mathcal{C}_i where $i \in [1, m]$, then there exist $i_1, i_2, i_3 \in [1, m]$ such that $\Theta_1(u_1)\Theta_1(u_2)$, $\Theta_2(u_1)\Theta_2(u_2)$ and $\Theta_3(u_1)\Theta_3(u_2)$, are in \mathcal{C}_{i_1} , \mathcal{C}_{i_2} and \mathcal{C}_{i_3} respectively, where i, i_1, i_2, i_3 are not necessary distinct.*

It is also important to note that in this step, the adjustment of any edge, say u_1u_2 , influences only the drawing of u_1u_2 within the “small neighborhoods” of both its ends, and that the adjustment does not change the relative topological positions between u_1u_2 and other edges in Γ_n . In particular, for $i = 1, 2$, we must pass through all the edges of $\mathcal{I}(u_i)$ in the same order in Γ_n^* as in Γ_n if we start from the fixed edge u_1u_2 and enumerate the edges of $\mathcal{I}(u_i)$ in clockwise, notwithstanding $\lambda_{\Gamma_n^*; u_i}(u_1u_2) \neq \lambda_{\Gamma_n; u_i}(u_1u_2)$.

Case 2. Suppose $n \geq 13$.

To obtain the desired drawing Γ_n^* for this case, we shall need the following assumption, which will be verified later in this subsection.

Assumption A. *For any vertex u in Γ_{n-2} , the edges $\mathcal{P}_1(u)\mathcal{P}_2(u)$, $\mathcal{P}_2(u)\mathcal{P}_3(u)$ and $\mathcal{P}_3(u)\mathcal{P}_4(u)$ are precisely drawn on the line $x = X_u$, and in particular,*

$$(|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_1(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_1(u))|) = (|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_4(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_4(u))|) = \left(\frac{n-1}{2}, \frac{n-1}{2}\right),$$

and that either

$$(|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_2(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_2(u))|) = (|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_3(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_3(u))|) = \left(\frac{n-3}{2}, \frac{n-1}{2}\right)$$

or

$$(|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_2(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_2(u))|) = (|\mathcal{A}_{\Gamma_n}^{-1}(\mathcal{P}_3(u))|, |\mathcal{A}_{\Gamma_n}^1(\mathcal{P}_3(u))|) = \left(\frac{n-1}{2}, \frac{n-3}{2}\right).$$

On the premise of Assumption A for any odd $n \geq 13$, we can get the desired drawing Γ_n^* by redrawing $\mathcal{P}_1(u)\mathcal{P}_2(u)$ and $\mathcal{P}_3(u)\mathcal{P}_4(u)$ to be arcs on the left plane of the line $x = X_u$ and redrawing $\mathcal{P}_2(u)\mathcal{P}_3(u)$ to be arcs on the right plane of the line $x = X_u$.

Step 2. Obtaining $\tilde{\Gamma}_{n+2}$ from Γ_n^* .

Take an arbitrary edge u_1u_2 in Γ_n^* . By (2) and (3), we can replace u_1u_2 by a bunch of four new edges $\{u_1^{(00)}u_2^{(00)}, u_1^{(10)}u_2^{(10)}, u_1^{(11)}u_2^{(11)}, u_1^{(01)}u_2^{(01)}\}$ in $\tilde{\Gamma}_{n+2}$ such that the “bunch” is drawn along the original route of u_1u_2 in Γ_n^* , with in particular,

$$\lambda_{\tilde{\Gamma}_{n+2}; u_i^{(x_1x_2)}}(u_1^{(x_1x_2)}u_2^{(x_1x_2)}) = \lambda_{\Gamma_n^*; u_i}(u_1u_2) \quad (14)$$

where $i = 1, 2$ and $x_1x_2 \in \{00, 10, 11, 01\}$.

In the meantime, the edges $u_i^{(00)}u_i^{(10)}$, $u_i^{(10)}u_i^{(11)}$, $u_i^{(11)}u_i^{(01)}$ are drawn precisely on the line $x = X_{u_i}$, and the edge $u_i^{(00)}u_i^{(01)}$ is drawn on the left plane or on the right plane of the line $x = X_{u_i}$ according to (12) or (13) holds for u_i in Γ_n^* respectively, where $i = 1, 2$.

Since every edge in Γ_n^* is Self-symmetric, it follows from (4), (6) and (14) that

$$\nu_{\tilde{\Gamma}_{n+2}}(\{u_1^{(00)}u_2^{(00)}, u_1^{(10)}u_2^{(10)}, u_1^{(11)}u_2^{(11)}, u_1^{(01)}u_2^{(01)}\}) = 0. \quad (15)$$

where u_1u_2 denotes an arbitrary edge in Γ_n^* .

Since Γ_n^* has Property 2, by (14) we derive the following.

Assertion B. For any vertex u in Γ_n^* , the drawing $\tilde{\Gamma}_{n+2}$ within the “small neighborhood” of the 4-cycle $\mathcal{P}_1(u)\mathcal{P}_2(u)\mathcal{P}_3(u)\mathcal{P}_4(u)$ coincides with $ML_{\frac{n+1}{2}}$ or $MR_{\frac{n+1}{2}}$.

Moreover, given an arbitrary Enclosed cycle $\mathcal{C} = u_0u_1 \cdots u_{k-1}$ in Γ_n^* for $n \in \{5, 7, 9\}$, by (4), (6), (8), (9), (10) and (11), we have that the two vertices $\mathcal{P}_{\mathcal{H}_\mathcal{C}^+(u_j)}(u_j)$ and $\mathcal{P}_{\mathcal{H}_\mathcal{C}^-(u_{j+1})}(u_{j+1})$ in $\tilde{\Gamma}_{n+2}$ are adjacent, where the subscripts of u are modulo k .

Consider the Fundamental structures in Γ_n^* again. We take a vertex subset $U = \{u_1, u_2, \dots, u_8\}$ of $V(Q_n)$ such that the drawing of $\mathcal{I}(U)$ forms a Fundamental structure in Γ_n^* , which is drawn without loss of generality as in (1) of Figure 3.2. We see that the edges of $E(\pi(U))$ in $\tilde{\Gamma}_{n+2}$ will

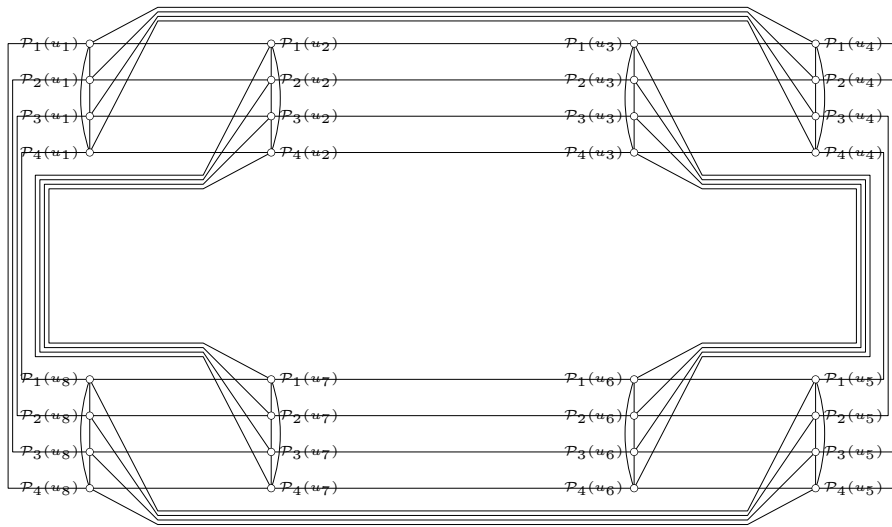


Figure 3.3: The drawing of $E(\pi(U))$ in $\tilde{\Gamma}_{n+2}$

be drawn as in Figure 3.3, where

$$\pi(U) = \bigcup_{i=1}^8 \{u_i^{(00)}, u_i^{(10)}, u_i^{(11)}, u_i^{(01)}\}.$$

Moreover, combined with (7) and (14), we have that

$$|\partial(\pi(U)) \cap \mathcal{A}_{\tilde{\Gamma}_{n+2}}^{-2}(\mathcal{P}_t(u_i))| = |\partial(\pi(U)) \cap \mathcal{A}_{\tilde{\Gamma}_{n+2}}^2(\mathcal{P}_t(u_i))| = \frac{n-3}{2} \quad (16)$$

for all $i \in [1, 8]$ and $t \in [1, 4]$.

The process of obtaining $\tilde{\Gamma}_7$ from Γ_5^* shown in Figure 3.5 will be helpful for us to understand accurately the adjustments in this step.

Step 3. Obtaining Γ_{n+2} from $\tilde{\Gamma}_{n+2}$.

To obtain the drawing Γ_{n+2} , we need to make two kinds of adjustments on the edges in $\tilde{\Gamma}_{n+2}$. The first kind is applied only for $n \in \{5, 7, 9\}$ and associated with the Enclosed cycles in Γ_n^* . The second is applied for all odd number $n \geq 5$ and associated with the Fundamental structures in Γ_n^* .

Suppose first $n \in \{5, 7, 9\}$. Let $\mathcal{C} = u_0 u_1 \cdots u_{k-1}$ denote an arbitrary Enclosed cycle of that decomposition of $V(Q_n)$ in Γ_n^* given in Property 4. For all $j \in [0, k-1]$, we adjust the drawing of the edge $\mathcal{P}_{\mathcal{H}_C^+(u_j)}(u_j) \mathcal{P}_{\mathcal{H}_C^-(u_{j+1})}(u_{j+1})$ within the “small neighborhoods” of *both* 4-cycles $\mathcal{P}_1(u_j) \mathcal{P}_2(u_j) \mathcal{P}_3(u_j) \mathcal{P}_4(u_j)$ and $\mathcal{P}_1(u_{j+1}) \mathcal{P}_2(u_{j+1}) \mathcal{P}_3(u_{j+1}) \mathcal{P}_4(u_{j+1})$, in particular, such that

$$\lambda_{\Gamma_{n+2};w}(e) \cap \{-1, 1\} = -(\lambda_{\tilde{\Gamma}_{n+2};w}(e) \cap \{-1, 1\})$$

and

$$\lambda_{\Gamma_{n+2};w}(e) \cap \{-2, 2\} = \lambda_{\tilde{\Gamma}_{n+2};w}(e) \cap \{-2, 2\} \quad (17)$$

where $e = \mathcal{P}_{\mathcal{H}_C^+(u_j)}(u_j) \mathcal{P}_{\mathcal{H}_C^-(u_{j+1})}(u_{j+1})$ and $w \in \{\mathcal{P}_{\mathcal{H}_C^+(u_j)}(u_j), \mathcal{P}_{\mathcal{H}_C^-(u_{j+1})}(u_{j+1})\}$. It is important to note that the four edges $u_j^{(00)} u_{j+1}^{(00)}$, $u_j^{(10)} u_{j+1}^{(10)}$, $u_j^{(11)} u_{j+1}^{(11)}$ and $u_j^{(01)} u_{j+1}^{(01)}$ are still parallel each other after the above adjustment of the edge $\mathcal{P}_{\mathcal{H}_C^+(u_j)}(u_j) \mathcal{P}_{\mathcal{H}_C^-(u_{j+1})}(u_{j+1})$. From Diagrams (3) and (4) of Figure 3.5, we can see the above adjustments, where the bold line means that the corresponding edge is the above adjusted edge “ $\mathcal{P}_{\mathcal{H}_C^+(u_j)}(u_j) \mathcal{P}_{\mathcal{H}_C^-(u_{j+1})}(u_{j+1})$ ”.

Since every vertex u in Γ_n^* belongs to exactly one Enclosed cycle, combined with Assertion B, we conclude that after the above kind of adjustment, the present drawing within the “small neighborhood” of the 4-cycle $\mathcal{P}_1(u) \mathcal{P}_2(u) \mathcal{P}_3(u) \mathcal{P}_4(u)$ coincides with one of $\widetilde{ML_{\frac{n+1}{2}}^T}$, $\widetilde{ML_{\frac{n+1}{2}}^B}$, $\widetilde{MR_{\frac{n+1}{2}}^T}$ and $\widetilde{MR_{\frac{n+1}{2}}^B}$.

Now we consider all odd number $n \geq 5$ and give the second kind of adjustments in this step. Let U be an arbitrary vertex subset of $V(Q_n)$ such that the drawing of $\mathcal{I}(U)$ forms a Fundamental structure in Γ_n^* , for which the corresponding drawing of $E(\pi(U))$ in $\tilde{\Gamma}_{n+2}$ is shown

as in Figure 3.3. Then we adjust the following edges

$$\begin{aligned} &\mathcal{P}_2(u_1)\mathcal{P}_2(u_4), \mathcal{P}_3(u_1)\mathcal{P}_3(u_4), \mathcal{P}_4(u_1)\mathcal{P}_4(u_4), \\ &\mathcal{P}_1(u_3)\mathcal{P}_4(u_6), \mathcal{P}_2(u_3)\mathcal{P}_3(u_6), \mathcal{P}_3(u_3)\mathcal{P}_2(u_6), \\ &\mathcal{P}_1(u_5)\mathcal{P}_1(u_8), \mathcal{P}_2(u_5)\mathcal{P}_2(u_8), \mathcal{P}_3(u_5)\mathcal{P}_3(u_8), \\ &\mathcal{P}_1(u_2)\mathcal{P}_4(u_7), \mathcal{P}_2(u_2)\mathcal{P}_3(u_7), \mathcal{P}_3(u_2)\mathcal{P}_2(u_7), \end{aligned}$$

which are shown as in Figure 3.4. Combined (16) and (17), we conclude that

$$\nu_{\Gamma_{n+2}}(E(\pi(U)), \partial(\pi(U))) = \nu_{\tilde{\Gamma}_{n+2}}(E(\pi(U)), \partial(\pi(U))),$$

and thus,

$$\nu_{\Gamma_{n+2}}(E(\pi(U)), E(Q_{n+2}) \setminus E(\pi(U))) = \nu_{\tilde{\Gamma}_{n+2}}(E(\pi(U)), E(Q_{n+2}) \setminus E(\pi(U))), \quad (18)$$

meanwhile, we can verify that

$$\nu_{\Gamma_{n+2}}(E(\pi(U))) = \nu_{\tilde{\Gamma}_{n+2}}(E(\pi(U))) - 8. \quad (19)$$

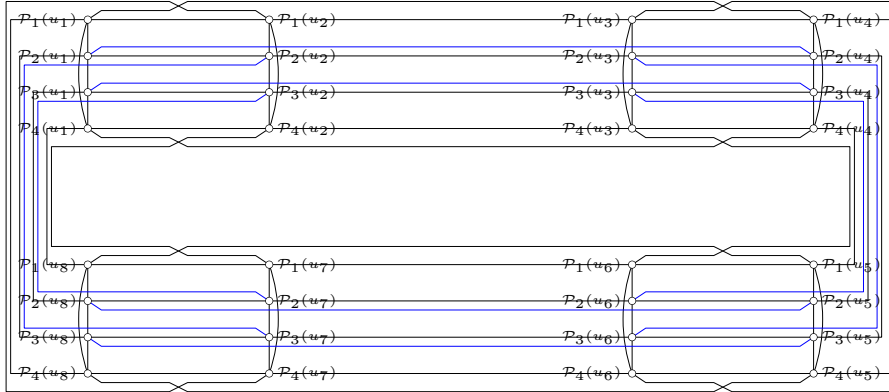


Figure 3.4: The drawing of $E(\pi(U))$ in Γ_{n+2}

Figure 3.5 and Figure 4.3 will be helpful for us to see the above two kinds of adjustment in this step, especially the first kind. In the diagrams for “ Γ_{n+2} ”, the bold line means that the corresponding edge is the adjusted edge associated with the Enclosed cycle in Γ_n^* .

To complete the whole process of obtaining Γ_{n+2} from Γ_n , it remains to verify that Assumption A holds for all odd $n \geq 13$. Recall that the kind of adjustment associated with the Enclosed cycle is applied only to $\tilde{\Gamma}_7, \tilde{\Gamma}_9, \tilde{\Gamma}_{11}$. Hence, combined with Assertion B, we can verify that Assumption A holds for all odd $n \geq 13$.

This completes the whole process obtaining Γ_{n+2} from Γ_n . Meanwhile, it is not hard to verify that the obtained drawing Γ_{n+2} has the desired two properties.

To make the whole process clearer, we also give the full drawings of Γ_7 and Γ_7^* in Figure 4.1 and Figure 4.2, respectively.

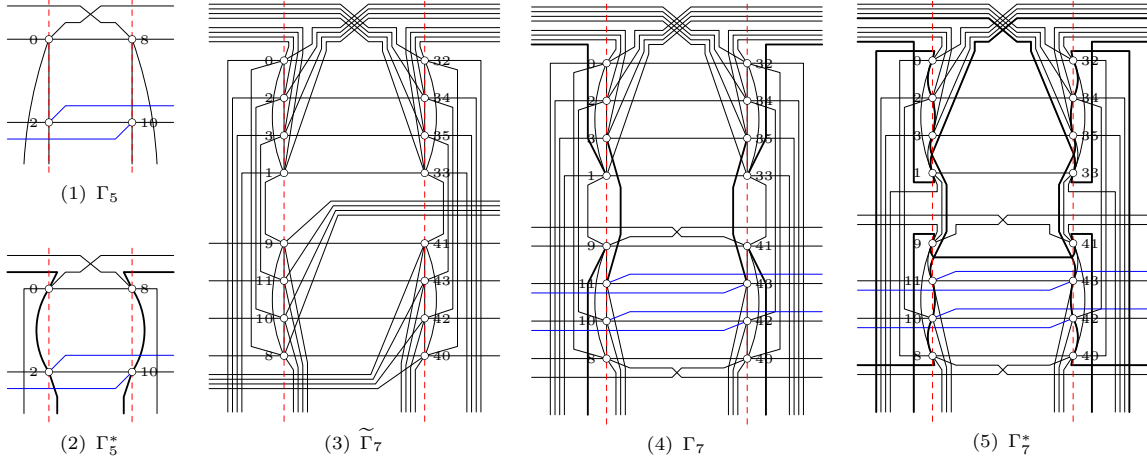


Figure 3.5: Auxiliary drawings illustrating the process from Γ_n to Γ_{n+2}

3.2 Construction of Γ_n for even n

The construction of the drawing Γ_n for all even number $n \geq 6$ is much easier than that for n is odd. However, the drawing Γ_n when n is even, is obtained from the drawing Γ_{n-1}^* given in subsection 3.1, for which the detailed process is follows.

- For convenience, we shall always denote n to be an even number no less than 6 in the rest of this subsection.

Let u be an arbitrary vertex in the drawing Γ_{n-1}^* . We locate the two new vertices $u^{(0)}$ and $u^{(1)}$ in Γ_n with

$$X_{u^{(0)}} = X_{u^{(1)}} = X_u,$$

$$Y_{u^{(0)}} = Y_u,$$

$$Y_{u^{(1)}} = Y_u + \frac{Y_{\hat{u}} - Y_u}{3},$$

and the new edge $u^{(0)}u^{(1)}$ precisely on the line $x = X_u$.

Let u_1u_2 be an arbitrary edge in the drawing Γ_{n-1}^* . We draw the two edges $u_1^{(0)}u_2^{(0)}$ and $u_1^{(1)}u_2^{(1)}$ in Γ_n to be a bunch such that the “bunch” is along the original route of u_1u_2 , in particular, with

$$\lambda_{\Gamma_n; u_i^{(x)}}(u_1^{(x)}u_2^{(x)}) = \lambda_{\Gamma_{n-1}^*; u_i}(u_1u_2)$$

where $i = 1, 2$ and $x \in \{0, 1\}$. Recalling that the drawing Γ_{n-1}^* has Property 1 and Property 2, similarly as in Subsection 3.1, we can derive that

$$\nu_{\Gamma_n}(\{u_1^{(0)}u_2^{(0)}, u_1^{(1)}u_2^{(1)}\}) = 0, \quad (20)$$

and that the following holds.

Assertion C. *The drawing Γ_n within the “small neighborhood” of the edge $u^{(0)}u^{(1)}$ coincides with $ML_{\frac{n}{2}}^2$ or $MR_{\frac{n}{2}}^2$, where u denotes an arbitrary vertex in Γ_{n-1}^* .*

3.3 Calculations of the number of crossings in Γ_n

In this subsection, we shall calculate the number of crossings in the drawing Γ_n constructed in Subsection 3.1 and Subsection 3.2. The following easy lemma will be useful for later calculations.

Lemma 3.1. *For any $k \in \mathbb{N}$ and $\delta \in \{2, 4\}$,*

$$\nu(MR_k^\delta) = \nu(ML_k^\delta) = \binom{\delta}{2} \cdot \binom{k}{2} + \binom{\delta}{2} \cdot \binom{k-1}{2} + (\delta-2) \cdot (k-1)$$

and

$$\nu(\widetilde{MR_k^T}) = \nu(\widetilde{MR_k^B}) = \nu(\widetilde{ML_k^T}) = \nu(\widetilde{ML_k^B}) = \nu(ML_k^4) - 1.$$

Combined the process of constructing the drawing Γ_n , Property 3, Property 4, (15), (18), (19), Assertion B and Lemma 3.1, we conclude that for all odd number $n \geq 5$,

$$\nu(\Gamma_n^*) = \nu(\Gamma_n),$$

$$\nu(\widetilde{\Gamma}_{n+2}) = 16 \cdot \nu(\Gamma_n^*) + 2^n \cdot \left(\binom{4}{2} \cdot \binom{\frac{n+1}{2}}{2} + \binom{4}{2} \cdot \binom{\frac{n-1}{2}}{2} + (n-1) \right)$$

and

$$\nu(\Gamma_{n+2}) = \nu(\widetilde{\Gamma}_{n+2}) - \epsilon_n \cdot 2^n - 8 \cdot 2^{\frac{n-3}{2}},$$

where

$$\epsilon_n = \begin{cases} 1, & \text{if } n \in \{5, 7, 9\}; \\ 0, & \text{if } n \geq 11, \end{cases}$$

which implies that

$$\begin{aligned} \nu(\Gamma_{n+2}) &= 16 \cdot \nu(\Gamma_n) + 2^n \cdot \left(\binom{4}{2} \cdot \binom{\frac{n+1}{2}}{2} + \binom{4}{2} \cdot \binom{\frac{n-1}{2}}{2} + (n-1) \right) - \epsilon_n \cdot 2^n - 8 \cdot 2^{\frac{n-3}{2}}. \end{aligned} \quad (21)$$

Similarly, by (20), Assertion C and Lemma 3.1, we can derive that for all even number $n \geq 6$,

$$\begin{aligned} \nu(\Gamma_n) &= 4 \cdot \nu(\Gamma_{n-1}^*) + 2^{n-1} \cdot \left(\binom{n/2}{2} + \binom{n/2-1}{2} \right) \\ &= 4 \cdot \nu(\Gamma_{n-1}) + 2^{n-3} \cdot (n-2)^2. \end{aligned} \quad (22)$$

In Figure 3.1, we see $\nu(\Gamma_5) = 56$. Then Theorem 1.1 follows from (21) and (22) immediately.

4 Concluding remarks

We first remark that for $n \in \{5, 7, 9\}$, the property of Conjoint-symmetric in constructing the drawing Γ_n^* is not necessary for such an construction but for the simplifications of composition as we said before. The interested readers may find other different adjustments to obtain the drawing Γ_n^* with the desired four properties.

Another thing authentically worthy mentioning is that the upper bound given in Theorem 1.1 can be improved slightly by still applying that kind of adjustments associated with the Enclosed cycles when obtaining the drawing Γ_{n+2} from $\tilde{\Gamma}_{n+2}$ for odd number $n \geq 11$. In fact, for $n \geq 11$ there exists the drawing of Q_n with Properties 1, 2 and 3 holding, in which we still can find several disjoint Enclosed cycles, say $\mathcal{C}_1, \dots, \mathcal{C}_m$, unfortunately, with $V(\mathcal{C}_1) \cup \dots \cup V(\mathcal{C}_m) \subsetneq V(Q_n)$. Theoretically, that would be insignificant since we have no general rule of finding those Enclosed cycles for all n . Indeed, the idea “*Enclosed cycles*” essentially is just an improved variant of the method employed in [6]. That is also why their construction can not be implemented inductively, even for $n = 9$.

We also want to note that, even without applying that kind of adjustments associated with the Enclosed cycles for $n \in \{5, 7, 9\}$ in subsection 3.1, we still get the drawing of Q_n with less crossings than the value conjectured by Erdős and Guy. The only difference is that the parameter, ϵ_n in (21), always takes zero.

Finally, we close this paper by a table making a comparison between the number of crossings in our drawing and the values conjectured by Erdős and Guy.

n	Conjectured values	Our results	Δ
5	56	56	0
6	352	352	0
7	1760	1744	16
8	8192	8128	64
9	35712	35424	288
10	151040	149888	1152
11	624128	619456	4672
12	2547712	2529024	18688
13	10311680	10238848	72832

Table 1: Comparison between the number of crossings in our drawing and the conjectured values

References

- [1] A.M. Dean, R.B. Richter, The crossing number of $C_4 \times C_4$, J. Graph Theory 19 (1995) 125–129.
- [2] R.B. Eggleton, R.K. Guy, The crossing number of the n -cube, Notices Amer. Math. Soc. 17 (1970) 757–757.
- [3] P. Erdős, R.K. Guy, Crossing number problems, Amer. Math. Monthly 80 (1973) 52–58.
- [4] F. Harary, Recent results in topological graph theory, Acta Math. Hungar. 15 (1964) 405–412.
- [5] L. Faria, C.M.H. de Figueiredo, On Eggleton and Guy’s conjectured upper bound for the crossing number of the n -cube, Math. Slovaca 50 (2000) 271–287.

- [6] L. Faria, C.M.H. de Figueiredo, O. Sýkora, I. Vrřo, An improved upper bound on the crossing number of the hypercube, *J. Graph Theory* 59 (2008) 145–159.
- [7] M.R. Garey, D.S. Johnson, Crossing number is NP-complete, *SIAM J. Alg. Disc. Math.* 4 (1983) 312–316.
- [8] T. Madej, Bounds for the crossing number of the n-cube, *J. Graph Theory* 15 (1991) 81–97.
- [9] F. Shahrokhi, O. Sýkora, L.A. Székely, I. Vrřo, The crossing number of a graph on a compact 2-manifold, *Adv. Math.* 123 (1996) 105–119.
- [10] P. Turán, A note of welcome, *J. Graph Theory* 1 (1977) 7–9.
- [11] W.T. Tutte, Toward a theory of crossing numbers, *J. Combinatorial Theory* 8 (1970) 45–53.

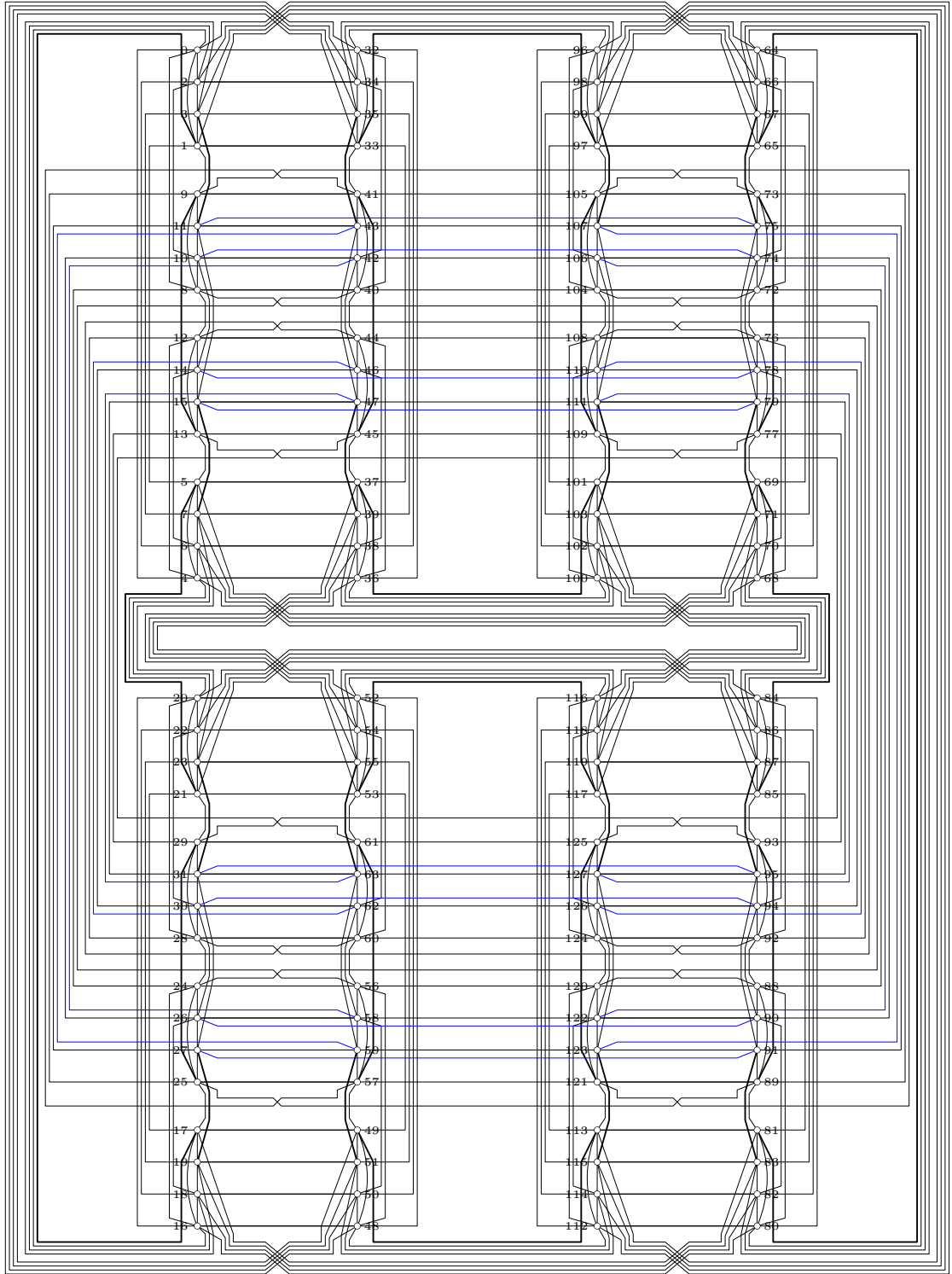


Figure 4.1: The full drawing of Γ_7

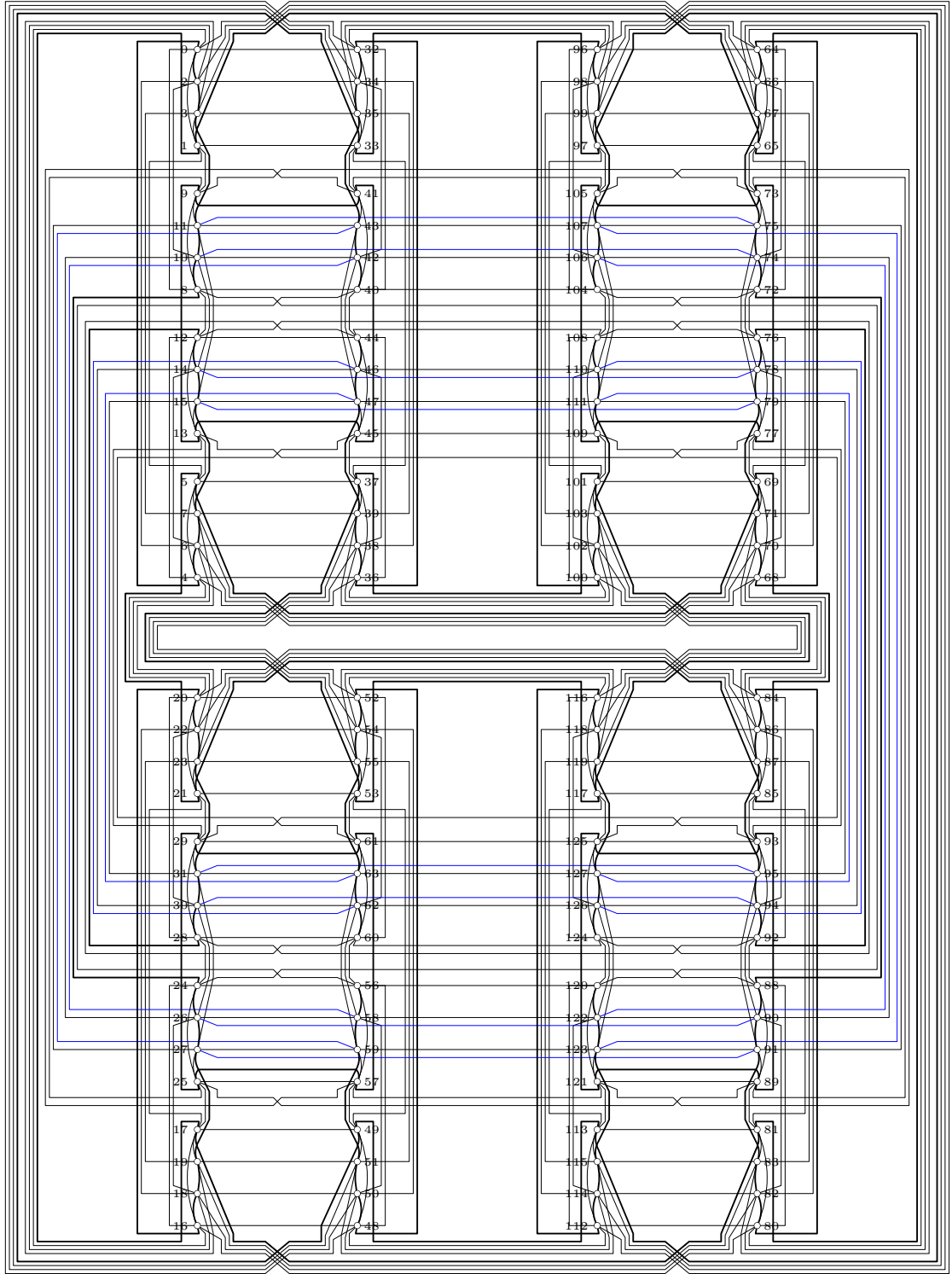
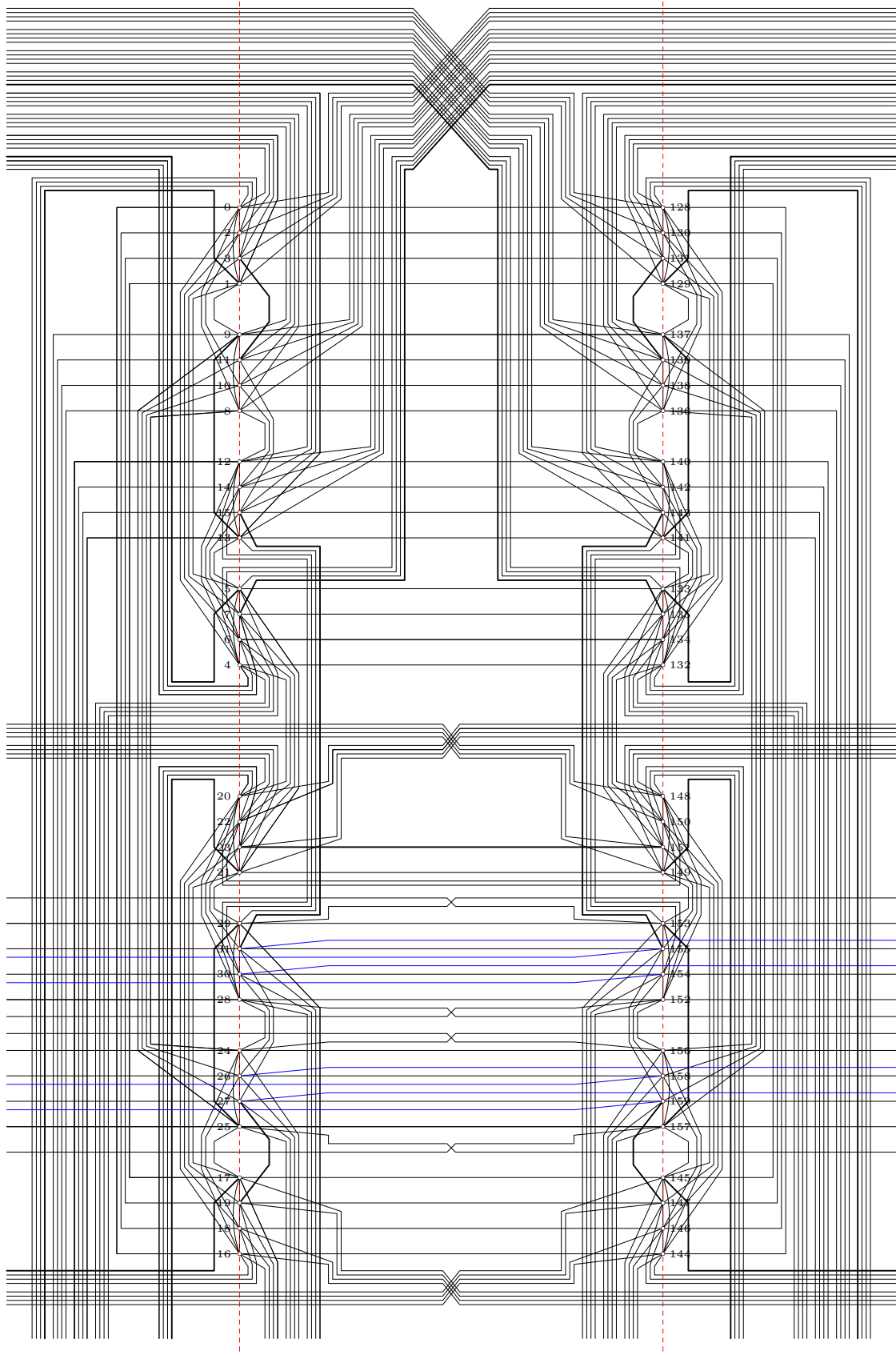
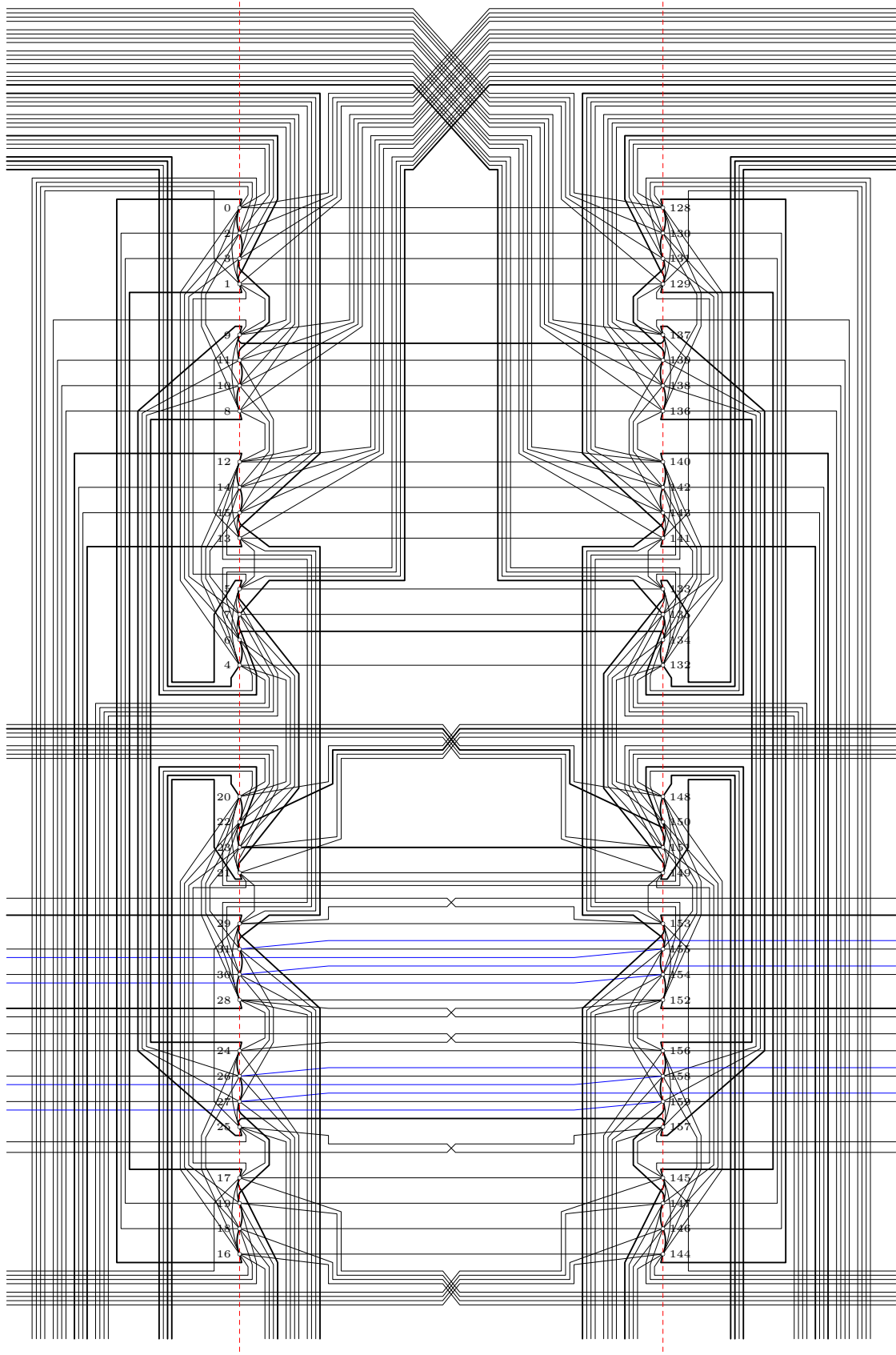


Figure 4.2: The full drawing of Γ_7^*



Γ_9

Figure 4.3: The partial drawing of Γ_9



Γ_9^*

Figure 4.4: The partial drawing of Γ_9^*